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Optimality Conditions for Fractional Minmax Programming

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1. INTRODUCTION

The purpose of this paper is to generalize the optimality conditions of Schmitendorf [6] and of fractional programming studied by Aggarwal and Saxena [1]. Aggarwal and Saxena considered the following problem:

$$\text{Minimize } F(x) = N(x)/D(x) = (f(x) + (x^t Bx)^{1/2})/h(x)$$
$$\text{subject to } g(x) \leqslant 0.$$

Where $x \in R^n$, B is an $n \times n$ symmetric positive semidefinite matrix, $f(\cdot)$, $-h(\cdot)$ are real valued, convex, differentiable functions defined on R^n and $g(\cdot)$ is a p-dimensional, convex, differentiable function also defined on R^n. The function $h(x) > 0$ for all feasible x.

We consider the following generalization of the Aggarwal–Saxena problem:

$$\begin{aligned} &\text{Minimize } F(x) = \sup_{y \in Y} (f(x, y) + (x^t Bx)^{1/2})/h(x, y) \\ &\text{subject to } g(x) \leqslant 0 \end{aligned} \tag{1}$$

where Y is a compact subset of R^m, $f(\cdot, \cdot)\colon R^n \times R^m \to R$ is C^1 on $R^n \times R^m$ and $g(\cdot)\colon R^n \to R^p$ is C^1 on R^n; B is an $n \times n$ positive semidefinite matrix; $h(\cdot, \cdot)\colon R^n \times R^m \to R$ is C^1 on $R^n \times R^m$. Throughout the paper we assume that $h(x, y) > 0$ for each (x, y) in $X \times Y$, where X is the set of feasible solutions of Problem (1), i.e., $X = \{x \in R^n\colon g(x) \leqslant 0\}$.

The necessary and sufficient conditions to be established are based on the optimality conditions developed by Schmitendorf [6] for a static minmax problem. This application is possible when the functions involved are differentiable at an optimal point or a candidate for optimal point. However, differentiability conditions may not hold due to the presence of $(x^t Bx)^{1/2}$ in the objective function. In this situation, the constraint qualification used by Aggarwal and Saxena and introduced by Mond [4] is extended for our case.

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As pointed out by Mond and Schechter [5], Mond's constraint qualification leads to the usual Kuhn–Tucker constraint qualification of classical nonlinear programming.

2. Preliminaries

We let X, the set of feasible solutions, be a compact set.

$$I(x^*) = \{i: g_i(x^*) = 0\}$$

$$Y(x) = \{y \in Y; (f(x, y) + (x^t Bx)^{1/2})/h(x, y) = \sup_{z \in Y}(f(x, z) + (x^t Bx)^{1/2})/h(x, z)\}$$

$K =$ the set of triplets $(s, t, \bar{y})$, where s ranges over the positive integers such that $1 \leqslant s \leqslant n+1$; $t = (t_1, ..., t_s)$ an s-dimensional vector with $t \geqslant 0$, $\sum_{i=1}^{s} t_i = 1$; $\bar{y} = (y_1, ..., y_s)$ an ms-dimensional vector with $y_i \in Y(x)$ $(i = 1, ..., s)$ for some $x \in R^n$.

For $x_0 \in X$, $(s, t, \bar{y}_0) \in K$, $y_0^i \in Y(x_0)$,

$$v_0 = (f(x_0, y_0^i) + (x_0^t Bx_0)^{1/2})/h(x_0, y_0^i),$$

we let

$$Q_{\bar{y}}(x_0) = \{z: z^t \nabla g_i(x_0) \leqslant 0 \ i \in I(x_0)$$

and

$$z^t \left(\sum_{i=1}^{s} (t_i/h(x_0, y_0^i))(\nabla_x f(x_0, y_0^i) - \nabla_x v_0 \cdot h(x_0, y_0^i)) \right) + z^t Bx_0/(x_0^t Bx_0)^{1/2} < 0 \qquad \text{if} \quad x_0^t Bx_0 > 0;$$

$$z^t \left(\sum_{i=1}^{s} (t_i/h(x_0, y_0^i))(\nabla_x f(x_0, y_0^i) - \nabla_x v_0 \cdot h(x_0, y_0^i)) \right) + (z^t Bz)^{1/2} < 0 \qquad \text{if} \quad x_0^t Bx_0 = 0 \Big\}.$$

We state the following two lemmas of Eisenberg [3] for easy reference.

Lemma 1 [3]. *Let B be a symmetric positive semidefinite matrix. Then $x^t Bz \leqslant (x^t Bx)^{1/2} (z^t Bz)^{1/2}$ for each x and z.*

LEMMA 2 [3]. *Let A be an $m \times n$ matrix and H be a symmetric positive semidefinite matrix. Let p be an n-dimensional vector. Then there exists an x such that $Ax \geqslant 0 \Rightarrow p^t x + (x^t H x)^{1/2} \geqslant 0$ if and only if there exist $u \in R^m$, $u \geqslant 0$, $w \in R^n$ such that $A^t u = Hw + p$, $Aw \geqslant 0$, $w^t H w \leqslant 1$.*

The next lemma is an extension of a result of Bhatia.

LEMMA [2]. *Suppose $x_0 \in S$, a compact subset of R^n. Let $y_0 \in Y(x_0)$. Then x_0 minimizes $(f(x, y_0) + (x^t Bx)^{1/2})/h(x, y_0)$ if and only if it minimizes $(f(x, y_0) + (x^t Bx)^{1/2}) - v_0 h(x, y_0))$, where $v_0 = \mathrm{Min}_{x \in S}(f(x, y_0) + (x^t Bx)^{1/2})/h(x, y_0)$.*

Proof. Suppose x_0 minimizes $(f(x, y_0) + (x^t Bx)^{1/2})/h(x, y_0)$ but it does not minimize $f(x, y_0) + (x^t Bx)^{1/2} - v_0 h(x, y_0)$. Let x_1 be such that it minimizes $f(x, y_0) + (x^t Bx)^{1/2} - v_0 h(x, y_0)$. Then $f(x_1, y_0) + (x_1^t Bx_1)^{1/2} - v_0 h(x_1, y_0) < f(x_0, y_0) + (x_0^t Bx_0)^{1/2} - v_0 h(x_0, y_0) = 0$. Therefore $(f(x_1, y_0) + (x^t Bx)^{1/2})/h(x_1, y_0) < v_0$. This means that v_0 is not the minimum value, which contradicts the hypothesis that it is. Hence x_0 also minimizes $f(x, y_0) + (x^t Bx)^{1/2} - v_0 h(x, y_0)$.

Conversely suppose x_0 minimizes $f(x, y_0) + (x^t Bx)^{1/2} - v_0 h(x, y_0)$ over S. But x_0 certainly minimizes $f(x, y_0) + (x^t Bx)^{1/2}/h(x, y_0)$ with its value equal to v_0. So the proof is complete.

3. OPTIMALITY CONDITIONS

The two theorems we are about to establish are direct generalizations of necessary and sufficient conditions of Schmitendorf [6]. (When $B = 0$, $h(x, y) = 1$, our theorems are the same as Schmitendorf's.) These results also generalize Theorems 4.1 and 4.2 of Aggarwal and Saxena [1] because of the presence of vector y in the objective function.

THEOREM 1 (Necessary Conditions). *If x_0 is an optimal solution of Problem 1 and $Q_{\bar{y}}(x_0)$ is empty, then there exist a positive integer s, $1 \leqslant s \leqslant n+1$, real vectors $t = (t_1, \ldots, t_s)$, $t \geqslant 0$, $u = (u_1, \ldots, u_p)$, $u \geqslant 0$, $w = (w_1, \ldots, w_n)$, $y_0^i \in Y(x_0)$ $(i = 1, \ldots, s)$ such that*

$$\sum_{i=1}^{s} (t_i/h(x_0, y_0^i))(\nabla_x f(x_0, y_0^i)) + Bw - v_0 \nabla_x h(x_0, y_0^i) + \sum_{i=1}^{p} u_i \nabla g_i(x_0) = 0$$

$$u_i g_i(x_0) = 0,\ i = 1, \ldots, p,\ w^t Bw \leqslant 1,\ (x_0^t Bx_0)^{1/2} = x_0^t Bw$$

and

$$\sum_{i=1}^{s} t_i + \sum_{i=1}^{p} u_i > 0 \qquad \textit{where} \quad v_0 = (f(x_0, y_0^i) + (x_0^t B x_0)^{1/2})/h(x_0, y_0^i),$$

$$y_0^i \in Y(x_0) \qquad \textit{for} \quad i = 1,..., s.$$

Proof. *Case* 1. Suppose $x_0^t B x_0 > 0$. The function $(f(x, y) + (xBx)^{1/2})/h(x, y)$ is differentiable with respect to x for each y. Hence by Schmitendorf's Theorem 1 [6], there exist a positive integer s, vectors $t \in R_+^s$, $u \in R_+^p$, $w \in R^n$, $y_0^i \in Y(x_0)$ $(i = 1,..., s)$ such that $u_i g_i(x_0) = 0$ $(i = 1,..., p)$, $\sum_{i=1}^{s} t_i + \sum_{i=1}^{p} u_i > 0$ and

$$\begin{aligned}\sum_{i=1}^{s} &((t_i/h(x_0, y_0^i))(\nabla_x f(x_0, y_0^i) + Bx_0)/(x_0^t B x_0)^{1/2}) \\ &- (t_i/(h(x_0, y_0^i))^2)(f(x_0, y_0^i) + (x_0^t B x_0)^{1/2} (\nabla_x h(x_0, y_0^i)) \\ &+ \sum_{i=1}^{p} u_i \nabla g_i(x_0) = 0. \end{aligned} \tag{2}$$

Letting $w = x_0/(x_0^t B x_0)^{1/2}$, (2) reduces to

$$\begin{aligned}\sum_{i=1}^{s} &(t_i/h(x_0, y_0^i))(\nabla_x f(x_0, y_0^i) + Bw - v_0 \nabla_x h(x_0, y_0^i)) \\ &+ \sum_{i=1}^{p} u_i \nabla g_i(x_0) = 0.\end{aligned}$$

Furthermore, since $w = x_0/(x_0^t B x_0)^{1/2}$, it follows that $w^t B w = 1$ and $(x_0^t B x_0)^{1/2} = x_0^t B w$. Therefore in this case all the conclusions of the Theorem hold.

Case 2. $x_0^t B x_0 = 0$. Since $Q_{\bar{y}}(x_0)$ is empty, $z^t \nabla g_i(x_0) \geqslant 0$ implies

$$z^t \sum_{i=1}^{s} (t_i/h(x_0, y_0^i))(\nabla_x f(x_0, y_0^i) - \nabla_x v_0 h(x_0, y_0^i)) + (z^t B z)^{1/2} \geqslant 0.$$

Hence by Lemma 2, there exist $u \geqslant 0$, $u \in R^{I(x_0)}$, $w \in R^n$ such that

$$u^t \nabla g_{I(x_0)}(x_0) + \sum_{i=1}^{s} (t_i/h(x_0, y_0^i))(\nabla_x f(x_0, y_0^i) - \nabla_x v_0 h(x_0, y_0^i)) + Bw = 0$$

and $w^t B w \leqslant 1$. Or

$$\sum_{i \in I(x_0)} u_i \nabla g_i(x_0) + \sum_{i=1}^{s} (t_i/h(x_0, y_0^i))(\nabla_x f(x_0, y_0^i) - \nabla_x v_0 h(x_0, y_0^i)) + Bw = 0$$

and $w^tBw \leqslant 1$. Since $g_i(x_0) = 0$ for $i \in I(x_0)$, $u_i g_i(x_0) = 0$ for $i \in I(x_0)$. Letting $u_i = 0$ for all i such that $g_i(x_0) < 0$, we have $u_i g_i(x_0) = 0$ $(i = 1,..., p)$. The fact that $u_i \geqslant 0$ and $(s, t, \bar{y}_0) \in K$ implies that $\sum_{k=1}^{s} t_i = 1$ and hence $\sum_{i=1}^{s} t_i + \sum_{i=1}^{p} u_i > 0$. From Lemma 1, $x_0^t B x_0 = 0$ implies $Bx_0 = 0$. Hence $(x_0^t B x_0)^{1/2} = 0 = x_0^t B w$. The proof is complete.

THEOREM 2 (Sufficient Conditions). *Suppose $g(\cdot)$ is a convex differentiable function, and $f(\cdot, y)$ and $-v_0 h(\cdot, y)$ are convex differentiable functions of x for each $y \in Y$. Suppose there exists a positive integer s, $1 \leqslant s \leqslant n+1$, real vectors $t \in R_+^s$, $u \in R_+^p$, $w \in R^n$ and $y_0^i \in Y(x_0)$, for $i = 1,..., s$, for some $x_0 \in X$ such that*

(i) $\sum_{i=1}^{s} (t_i/h(x_0, y_0^i))(\nabla_x f(x_0, y_0^i) - \nabla_x v_0 h(x_0, y_0^i) + Bw) + \sum_{i=1}^{p} u_i \nabla g_i(x_0) = 0$,

(ii) $u_i g_i(x_0) = 0$ *for* $i = 1,..., p$,

(iii) $w^tBw \leqslant 1$, $(x_0^t B x_0)^{1/2} = x_0^t B w$, $\sum_{i=1}^{s} t_i > 0$. *where* $v_0 = (f(x_0, y_0^i) + (x_0^t B x)^{1/2})/h(x_0, y_0^i)$ *for* $i = 1,... s$. *Then x_0 is an optimal solution of Problem* (1).

Proof. Suppose x_0 is not an optimal solution of (1). Then there exists an $x_1 \in X$ such that

$$\sup_{y \in Y}((f(x_1, y) + (x_1^t B x_1)^{1/2})/h(x_1, y)) < \sup_{y \in Y}((f(x_0, y) + (x_0^t B x_0)^{1/2})/h(x_0, y)). \tag{3}$$

Also

$$u_i g_i(x_0) = 0, \qquad i = 1,..., p \tag{4}$$

$$u_i g_i(x_1) \leqslant 0, \qquad i = 1,..., p. \tag{5}$$

We note that

$$\sup_{y \in Y}((f(x_0, y) + (x_0^t B x_0)^{1/2})/h(x_0, y)) = (f(x_0, y_0^i) + (x_0^t B x_0)^{1/2})/h(x_0, y_0^i) \tag{6}$$

for $i = 1,..., s$ and

$$(f(x_1, y_0^i) + (x_1^t B x_1)^{1/2})/h(x_1, y_0^i) \leqslant \sup_{y \in Y}((f(x_1, y) + (x_1^t B x_1)^{1/2})/h(x_1, y)). \tag{7}$$

Hence by (3), (6) and (7),

$$(f(x_1, y_0^i) + (x_1^t B x_1)^{1/2})/h(x_1, y_0^i) < (f(x_0, y_0^i) + (x_0^t B x_0)^{1/2})/h(x_0, y_0^i) \tag{8}$$

for $i = 1,\dots, s$. By hypothesis (i),

$$\sum_{i=1}^{s} (t_i/h(x_0, y_0^i))(x_1 - x_0)^t \, (\nabla_x f(x_0, y_0^i) + Bw - v_0 \nabla_x h(x_0, y_0^i))$$
$$= -\sum_{i=1}^{p} u_i (x_1 - x_0)^t \, \nabla g_i(x_0). \tag{9}$$

Since g_i is convex for each i,

$$\sum_{i=1}^{p} (u_i \, g_i(x_1) - u_i \, g_i(x_0)) \geqslant \sum_{i=1}^{p} u_i (x_1 - x_0)^t \, \nabla g_i(x_0). \tag{10}$$

By (4), (5) and (10), we have

$$\sum_{i=1}^{p} u_i (x_1 - x_0)^t \, \nabla g_i(x_0) \leqslant 0. \tag{11}$$

By (9) and (11),

$$\sum_{i=1}^{s} (t_i/h(x_0, y_0^i))((x_1 - x_0)^t \, (\nabla_x f(x_0, y_0^i) - v_0 \nabla_x h(x_0, y_0^i))$$
$$+ x_1^t B w - x_0^t B w) \geqslant 0. \tag{12}$$

Therefore by the convexity of f and $-h$ with respect to x for each y and using (12),

$$\sum_{i=1}^{s} (t_i/h(x_0, y_0^i))(f(x_1, y_0^i) - f(x_0, y_0^i) + v_0 h(x_0, y_0^i)$$
$$- v_0 h(x_1, y_0^i) + x_1^t B w - x_0^t B w) \geqslant 0. \tag{13}$$

Since $v_0 = (f(x_0, y_0^i) + (x_0^t B x_0)^{1/2})/h(x_0, y_0^i)$ for $i = 1,\dots, s$ and by Lemma 1, (13) becomes

$$\sum_{i=1}^{s} (t_i/h(x_0, y_0^i))(f(x_1, y_0^i) + (x_0^t B x_0)^{1/2} - (f(x_0, y_0^i)$$
$$+ (x_0^t B x_0)^{1/2})/h(x_0, y_0^i)) \cdot h(x_1, y_0^i) + (x_1^t B x_1)^{1/2} \, (w^t B w)^{1/2} - x_0^t B w \geqslant 0. \tag{14}$$

Now by hypothesis (iii), (14) reduces to

$$\sum_{i=1}^{s} (t_i/h(x_0, y_0^i))(f(x_1, y_0^i) + (x_1^t B x_1)^{1/2} - (f(x_0, y_0^i) + (x_0^t B x_0)^{1/2})/h(x_0, y_0^i)) - h(x_1, y_0^i) \geqslant 0.$$

Or

$$\sum_{i=1}^{s} ((t_i/h(x_1, y_0^i)/h(x_0, y_0^i))((f(x_1, y_0^i) + (x_1^t B x_1)^{1/2})/h(x_1, y_0^i)) - (f(x_0, y_0^i) + (x_0^t B x_0)^{1/2})/h(x_0, y_0^i)) \geqslant 0.$$

But this last inequality is impossible since $t_i \geqslant 0$, $h(x_0, y_0^i) > 0$, $h(x_1, y_0^i) > 0$ and

$$((f(x_1, y_0^i) + (x_1^t B x_1)^{1/2})/h(x_1, y_0^i)) - (f(x_0, y_0^i) + (x_0^t B x_0)^{1/2})/h(x_0, y_0^i) < 0 \qquad \text{by (8).}$$

This contradiction leads to the conclusion that x_0 minimizes Problem (1).

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